# $A_\infty$-Category Theory

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# Chapter 1

# Introduction

This project formalizes the definition of $A_\infty$-categories and constructs an $A_\infty$-endofunctor on a variant of the KLRW category, which acts as a braid group invariant.

# Chapter 2

# Gradings

**Definition 1** (Parity group)**.** The type of parities is $\mathbb{Z}/2\mathbb{Z}$, implemented as `ZMod 2`.

**Definition 2** (Grading)**.** A grading on a type $\beta$ consists of an additive commutative group structure on $\beta$, together with homomorphisms

$$\phi : \mathbb{Z} \to \beta, \qquad \sigma : \beta \to \mathbb{Z}/2\mathbb{Z},$$

such that $\sigma(\phi(n))$ is the parity class of $n$ for every integer $n$.

**Definition 3** (Integer shift)**.** For a grading index $\beta$, is the image of $n \in \mathbb{Z}$ under the distinguished map $\phi : \mathbb{Z} \to \beta$.

**Definition 4** (Graded $R$-module)**.** For a commutative ring $R$, a graded $R$-module is a $\beta$-indexed family of $R$-modules, implemented as a graded object in `ModuleCat R`.

**Definition 5** ($R$-linear graded quiver)**.** An $R$-linear graded quiver on an object type Obj assigns to each pair of objects $(X, Y)$ a graded $R$-module of morphisms from $X$ to $Y$.

# Chapter 3

# Stasheff Data

**Definition 6** (Operation target degree)**.** Given input degrees $(d_0, \dots, d_{n-1})$, the output degree of the $n$-ary operation is

$$\sum_i d_i + (2 - n),$$

where the final summand is interpreted using the grading shift .

**Definition 7** (Stasheff target degree)**.** The common degree of the arity-$n$ Stasheff relation is

$$\sum_i d_i + (3 - n)$$

in the grading index.

**Definition 8** (Composable morphism type)**.** For a string of objects and prescribed degrees,  is the $i$-th graded hom-module in that composable string.

**Definition 9** (Operation target type)**.** For a composable string of objects and degrees, this is the graded hom-module in which the corresponding multilinear operation takes values.

**Definition 10** (Indexed Stasheff term)**.** Fix object-indexed operations $m_n$. For valid indices $(r, s)$,  is the term obtained by first applying the inner $s$-ary operation in positions $r, \dots, r + s - 1$ and then applying the outer operation to the collapsed string.

**Definition 11** (Stasheff sign parity)**.** For a valid pair $(r, s)$, this is the parity

$$|a_{r+s}| + \cdots + |a_{n-1}| - (n - r - s)$$

computed in .

**Definition 12** (Stasheff sign)**.** The integer sign attached to $(r, s)$ is $(-1)$ raised to the parity recorded by .

**Definition 13** (Indexed Stasheff sum)**.** The arity-$n$ Stasheff sum is the signed sum of all valid composites of an inner operation followed by an outer operation.

**Definition 14** (Indexed Stasheff identities)**.** An object-indexed family of multilinear operations satisfies the Stasheff identities if every indexed Stasheff sum vanishes.

# Chapter 4

# A-infinity Categories

**Definition 15** ($A_\infty$-precategory)**.** An $A_\infty$-precategory over $R$ with object type Obj consists of a graded $R$-linear quiver together with operations

$$m_n$$

of the prescribed multilinear type for every positive arity $n$.

## 4.1 Chains

**Definition 16** (Chain)**.** A chain in an $A_\infty$-precategory consists of a positive length $n$, a string of $n + 1$ objects, and a degree assigned to each of the $n$ composable morphism slots.

**Definition 17** (Chain source)**.** The source of a chain is its initial object.

**Definition 18** (Chain target)**.** The target of a chain is its final object.

**Definition 19** (Chain link type)**.** For a chain and an index $i$, is the module of the $i$-th composable morphism in that chain.

**Definition 20** (Chain operation target)**.** For a chain, this is the graded hom-module that receives the corresponding multilinear operation.

## 4.2 Stasheff Identities

**Definition 21** (Precategory Stasheff sum)**.** The Stasheff sum of an $A_\infty$-precategory is the indexed Stasheff sum formed from its structure maps.

**Definition 22** (Precategory Stasheff property)**.** An $A_\infty$-precategory satisfies the Stasheff identities when its structure maps define an object-indexed system satisfying .

**Definition 23** ($A_\infty$-category)**.** An $A_\infty$-category is an $A_\infty$-precategory together with a proof that its operations satisfy the Stasheff identities.

**Lemma 24** (Stasheff sum vanishes in an $A_\infty$-category)**.** *For an $A_\infty$-category, every Stasheff sum computed from the structure maps is equal to zero.*

# Chapter 5

# A-infinity Algebras

**Definition 25** ($A_\infty$-algebra structure)**.** Let $A$ be a graded $R$-module. An $A_\infty$-algebra structure on $A$ consists of multilinear operations

$$m_n : A(d_0) \otimes \cdots \otimes A(d_{n-1}) \to A\left(\sum_i d_i + 2 - n\right)$$

for every positive arity $n$ and every choice of input degrees.

**Definition 26** (Algebra to precategory)**.** Any $A_\infty$-algebra structure determines a one-object $A_\infty$-precategory, with object type  and constant hom-module equal to the underlying graded module.

**Definition 27** (Algebra Stasheff sum)**.** The Stasheff sum of an $A_\infty$-algebra is the Stasheff sum of its associated one-object $A_\infty$-precategory.

**Definition 28** (Algebra Stasheff property)**.** An $A_\infty$-algebra structure satisfies the Stasheff identities when every one of its Stasheff sums is zero.

**Definition 29** ($A_\infty$-algebra)**.** An $A_\infty$-algebra is an $A_\infty$-algebra structure together with a proof of the Stasheff identities.

**Lemma 30** (Stasheff sum vanishes in an $A_\infty$-algebra)**.** *For an $A_\infty$-algebra, every algebraic Stasheff sum is equal to zero.*

**Definition 31** ($A_\infty$-algebra to precategory)**.** An $A_\infty$-algebra canonically determines a one-object $A_\infty$-precategory.

# Chapter 6

# KLRW Category

## 6.1 Overview

## 6.2 Computable Add

**Definition 32.** Let $\mathcal{C}$ be a preadditive category. Then the *computable additive completion* of $\mathcal{C}$, denoted CMat($\mathcal{C}$), has

- as objects, lists of elements in $\mathcal{C}$.
- as morphisms, dependent matrices of morphisms in $\mathcal{C}$. Specifically, if $(A_0, A_1, \ldots, A_n), (B_0, B_1, \ldots, B_m) \in \mathrm{Ob}\,(\,\mathrm{Mat\_}(\mathcal{C}))$, then

$$\mathrm{Hom}\,((A_0, A_1, \ldots, A_n), (B_0, B_1, \ldots, B_m)) = \left\{ \begin{bmatrix} a_{00} & \cdots & a_{0,n-1} \\ \vdots & \ddots & \vdots \\ a_{m-1,0} & \cdots & a_{m-1,n-1} \end{bmatrix} : a_{ij} \in \mathrm{Hom}(A_j, B_i) \right\}$$

  where $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$.
- composition is defined by matrix multiplication
- identities are given by the identity matrix

$$(\mathbb{1}_{(A_0,\ldots,A_m)})_{i,j} = \begin{cases} \mathbb{1}_{A_i} & \text{reinterpreted as an element of } \mathrm{Hom}(A_i, A_j) \text{ by casting along the equality if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

.

There is a fully faithful functor $G : \mathcal{C} \Rightarrow \mathrm{CMat\_}(\mathcal{C})$.

There is a fully faithful embedding functor $F : \mathrm{CMat\_}(\mathcal{C}) \Rightarrow \mathrm{Mat\_}(\mathcal{C})$. It is (nonconstructively) essentially surjective, so $F$ is an equivalence of categories.

**Definition 33.** Given $A = (A_0, \ldots, A_n), B = (B_0, \ldots, B_n) \in \mathrm{CMat\_}(\mathcal{C})$, the *computable biproduct* of $A$ and $B$ is defined by $A \boxplus_k B = (A_0, \ldots, A_n, B_0, \ldots, B_m)$

**Definition 34.** For any $Z \in \mathrm{CMat\_}(\mathcal{C})$, the *index type*, denoted $\iota_Z$, is the type $\{0, \ldots, (\mathrm{len}(Z) - 1)\}$, and the indexing function, $X_2 : \iota_Z \to \mathcal{C}$, is defined by $X_Z(i) = Z_i$.

**Definition 35.** Let $n \in \mathbb{N}$. A *positioning* of 1 black strand with $n$ red strands is an element of $\{0, \ldots, n+1\}$. Given positionings of $X$ and $Y$, the strand set $StrandSpace_{X,Y} = \mathbb{N}$, representing the number of dots. (Note here that $\text{StrandSpace}_{X,Y}$ does not depend on $X$ or $Y$, but with more black strands it might).

**Definition 36.** If $X, Y, Z$ are positionings of 1 black strand and $n$ red strands and $a \in \text{StrandSpace}_{X,Y}$, $b \in \text{StrandSpace}_{Y,Z}$, then their composition is given by

$$\begin{aligned} b \circ a &= a + b + \frac{|X - Y| + |Y - Z| - |X - Z|}{2} \\ &= a + b + \max(x, y) + \max(y, z) - \max(x, z) - y \end{aligned}$$

**Definition 37.** Fix a commutative ring $R$. The KLRW category of 1 black strand and $n$ red strands, denoted KLRW category of 1 black strand and $n$ red strands, denoted $\text{KLRW}_n^R$, is given by

- $\text{Ob}(\text{KLRW}_n^R)$ is the set of positionings of 1 black strand with $n$ red strands
- $\text{Hom}(X, Y)$ is the free $R$-module generated by $\text{StrandSpace}_{X,Y}$.

Composition in $\text{KLRW}_n^R$ is defined by extending the composition of elements of the strand space linearly.

Since each $\text{Hom}(X, Y)$ is an $R$-module and composition is $R$-linear, $\text{KLRW}_n^R$ is $R$-linear and thus preadditive.

**Definition 38.** Let $\mathcal{C}$ be a preadditive category with a zero object. Then a bounded cochain complex of $\mathcal{C}$ is a ($\mathbb{Z}$-graded) cochain complex $A^\bullet$ of $\mathcal{C}$ equipped with a finite set $S$ called the support, satisfying $\forall i \in \mathbb{Z}, i \in S \Leftrightarrow A^i \ncong 0$. Note that $S$ is uniquely determined by $A^\bullet$, though included for computability.

## 6.3 Bounded Cochain Complexes

**Definition 39.** Let $\text{BK}^\bullet(\mathcal{C})$ denote the set of bounded cochain complexes in $\mathcal{C}$. THen there is a function $F : \text{BK}^\bullet(\mathcal{C}) \to K^\bullet(\mathcal{C})$ which forgets the finiteness. Then we equip $\text{BK}^\bullet(\mathcal{C})$ with the induced category structure of $F$ which makes $F$ into a fully faithful functor $F : \text{BK}^\bullet(\mathcal{C}) \to K^\bullet(\mathcal{C})$.

## 6.4 $\beta$-functor

### 6.4.1 Functor data

We will now specify the data needed to define how a braiding functor acts. Let $R, S, T \in \mathcal{KLRW}$.

The data of a braiding functor is given by the following functions:

$$\begin{aligned} \beta_0 &: & \mathcal{KLRW} &\to K^* \\ \beta_1 &: & \text{Hom}(R, S) &\to \text{Hom}(\beta_0(R), \beta_0(S)) \\ \beta_2 &: & \text{Hom}(R, S) \times \text{Hom}(S, T) &\to \text{Hom}(\beta_0(R), \beta_0(T)) \end{aligned}$$

In addition to these functions, there must also be proofs that they satisfy $A_\infty$ relations (to be specified).

One can understand these functions as specifying how the functor acts on the generating elements of our chain complex category. The action of the functor is as follows.

### 6.4.2 Functor action

Firstly, we may extend the functions acting on $\mathcal{KLRW}$ to act on $\mathrm{Add}(\mathcal{KLRW})$ in a linear sense. We define:

$$\hat{\beta}_0\colon \mathrm{Add}(\mathcal{KLRW}) \to K^*$$

by setting:

$$\hat{\beta}_0\left(\bigoplus_i S_i\right) = \bigoplus_i \beta_0(S_i)$$

Extending the remaining two functions requires more care.

For $\beta_1$, let $\bigoplus_i A_i$ and $\bigoplus_j B_j$ be objects in $\mathrm{Add}(\mathcal{KLRW})$. Let $f \in \mathrm{Hom}\left(\bigoplus_i A_i, \bigoplus_j B_j\right)$. For each pair $k, l$, we project $f$ to its component $f_{k,l} \in \mathrm{Hom}(A_k, B_l)$. We now create $g_{k,l} \in \mathrm{Hom}\left(\hat{\beta}_0\left(\bigoplus_i A_i\right), \hat{\beta}_0\left(\bigoplus_j B_j\right)\right)$ by extending this morphism. Formally, $g_{k,l}$ is constructed by composing the canonical projection, the mapped component, and the canonical injection:

$$\hat{\beta}_0\left(\bigoplus_i A_i\right) \twoheadrightarrow \beta_0(A_k) \xrightarrow{\beta_1(f_{k,l})} \beta_0(B_l) \hookrightarrow \hat{\beta}_0\left(\bigoplus_j B_j\right)$$

For $\beta_2$, we construct the extension $\hat{\beta}_2$ analogously.

Note now that if $A^* \in K^*$, then we may write $A^*$ as the cone of the following chain map:

$$\begin{array}{ccccccccc} 0 & \to & A_0 & \to & \cdots & \to & A_{n-1} & \to & 0 \\ & & \downarrow & & & & \downarrow & & \\ 0 & \to & 0 & \to & \cdots & \to & A_n & \to & 0 \end{array}$$

We may call the upper chain complex $A^*_{[0,n-1]}$, and the lower $A^*_{[n,n]}[-1]$ (shifted by $-1$ because $A_{n-1}$ and $A_n$ need the same homological grading).

Assuming we know how $\overline{\beta}_0$ acts on these two chain complexes, we will state how it acts on $A^*$. First, take the direct sum:

$$\overline{\beta}_0(A^*) = \overline{\beta}_0(A^*_{[0,n-1]}) \oplus \overline{\beta}_0(A^*_{[n,n]}[-1])$$